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The entropy formula of information theory measures, in the limit, the minimum expected number of symbols needed to store or transmit a long sequence of decodable messages. It is not related to other quantities, also relevant to the cost, or relevant to the benefit, of information. And if it is at all useful to compare degrees of "uncertainty", any concave symmetric function on probability space has the "intuitively" desirable properties.

The number of required symbols, e.g., of specified "parameters", does measure "disorder", said to characterize physical entropy. However, in contrast to the minimum expected message length, the entropy of statistical physics is not derived by extremization; it is, in the limit, related to the probability of a given allocation of states among a large number of entities.

The most probable allocation (and thus maximum entropy) which implies, in physics, the basic relation between temperature and the changes of heat and entropy, has been sometimes interpreted as the most probable income distribution. Maximization of entropy has also been proposed for statistical inference, without clear justification.

All this must be distinguished from the study of physical entropy in organized human agglomerations.

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ENTROPY, ECONOMICS, PHYSICS

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ENTROPY, ECONOMICS, PHYSICS

Jacob Marschak, University of California at Los Angeles *)

I. As pointed out by C. Brumat, no extremization occurs in the physicists' (e.g. E. Schroedinger's) derivation of entropy. With $p =_{df} (p_1, \ldots, p_m)$,

$$H(p) =_{df} - \sum_1^m p_i \ln p_i =$$

$$= \lim_{N \to \infty} \{[\ln P(p;N)]/N\} + \ln m \; ,$$

where $P(p;N) =_{df}$ Probability that Np_i entities are in state i $(i=1,\ldots,m)$ $= [N!/\prod_1^m (Np_i)!] \cdot \prod_1^m \pi_i^{Np_i}$;

provided prior uniformity is assumed, i.e., $\pi_i = 1/m$, all i.

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II. By contrast, in "<u>information theory</u>", entropy is derived by extremization:

$$H(p) = \lim_{b\to\infty} (\min_{w_1,\ldots,w_m} \sum_1^m p_i w_i \cdot \ln r)/b$$

subject to the decodability condition

$$\Sigma_r^{-w_i} \leq 1 \;;$$

where w_i= number of digits in the code word encoding the i-th sequence ("block") of b states; and r= size of code alphabet. H(p) thus does measure "disorder": the description of a crystal, a circle can be encoded in fewer parameters than a scatter of points. But do physicists relate entropy to disorder <u>via</u> efficient coding?

III. Thus H increases with the length of an economically coded message, and hence with the expected cost

<u>of storing and transmitting information</u>; but <u>not</u> with the <u>cost of collecting</u> it; nor, given the cost, with the expected <u>gain</u> to the information user (as seems to be implied by H. Theil?). The latter depends, not only on p, but also on the "benefit function" β (of actions a_j and states z_i). In particular, the expected gain from <u>perfect</u> information about the z_i is

$$g_\beta(p) =_{df} \sum_i p_i \max_j \beta(a_j, z_i) - \max_j \sum_i p_i \beta(a_j, z_i) \geq 0 \;.$$

The function $g_\beta(\cdot)$ is concave in p but not necessarily symmetric. (The same is true of imperfect information).

IV. "Intuitively", for any "uncertainty function" U(p):

(a) U is symmetric;

(b) $U(p) \leq U(1/m,\ldots,1/m)$

(c) perfect information

$=_{df} U(p) - U(1,0,\ldots,0)$ is non-negative.

Now, these properties are shared by H with all other symmetric quasi-concave functions (or, extending (c) to imperfect information, with all symmetric concave functions: De Groot). As to the additivity property

(d) $H(p,q) = H(p) + H(q)$ for p,q independent:

it is exclusive to H but is relevant only to message length. (Note: transportation and ware-housing costs, too, are additive and are independent of the benefits and production costs of goods moved and stored!). To compare and order (rather than to measure), additivity is not required. W. Hildenbrand, and H. Paschen, H. Theil, and others, used entropy to compare degrees of "concentration" (e.g., of an industry) and were criticized by P. Hart on empirical grounds.

V. Now define

$$X =_{df} \sum_{1}^{m} p_i x_i = \text{fixed average } \underline{\text{income}}\text{; and,}$$

$$\text{with } x =_{df} (x_1, \ldots, x_m) \text{ fixed,}$$

$$H^*(X,x) =_{df} H(p^{(X,x)}) =_{df} \max_p H(p)$$

$$\text{subject to } \Sigma p_i x_i = X.$$

Then (with a Lagrange λ depending on X,x)

$$p_i^{(X,x)} = e^{-\lambda x_i} / \text{const.}$$

Such an <u>income distribution</u> (F.P. Cantelli) is hardly realistic (prior uniformity was assumed in I. above!). But, if X is interpreted as average <u>energy</u> and λ as inversely proportional to absolute <u>temperature</u> T, the defining property of the original, non-stochastic, concept of physical entropy obtains:

$$dH^*(X,x) = dQ/T, \quad \text{where}$$

$$dQ =_{df} dX - \Sigma\, p_i\, dx_i =_{df} \text{heat supply.}$$

Detailed analogies with both information theory and a theory of income distribution have been proposed by H. Reiss.

VI. The constrained maximization of H, as above, was proposed by E. Jaynes and M. Tribus for the case of limited knowledge of prior probabilities and was recently applied to stock market analysis (J. Cozzolino and M. Zahner; D. Griffith and J. Snell). This approach should be compared with the "Laplace" assumption of uniformity over the probability space, whether constrained with respect to its dimensionality n of $p = (p_1, \ldots, p_m)$ or to some other property. Examples:

Constraint:	Estimate of p: Laplace	Jaynes
m=2.	(1/2,1/2)	(1/2,1/2)
m=2; $p_1 \geq 2/3$.	(5/6,1/6)	(2/3,1/3)
m=3; $p_1 \leq 2/3$.	(1/3,2/3)	(1/2,1/2)
m=3.	(1/3,1/3,1/3)	(1/3,1/3,1/3)
m=3; $p_2 = p_3$	(1/2,1/4,1/4)	(1/3,1/3,1/3)
m=3; $p_1 + 2p_2 + 4p_3 = 2$	(1/3,1/2,1/6)	(.43,.35,.22) approx.

The point in the (constrained) probability space that is obtained by the "Laplace" approach can be regarded as the mean of the distribution of probability distributions. Superficially, the "Jaynes" approach might be said to determine its mode, and thus have the advantage of invariance under transformations of the random variable. However, as stated at the beginning of this paper, entropy is related

to the probability of a given distribution under the assumption of prior uniformity over the set of values (called "states") of the random variable: $\pi_i = 1/m$. This removes the invariance of the mode.

VII. In another context of statistical inference, maximizing a generalized entropy amount ("discrimination information") was recommended by S. Kullback and has been applied to econometric estimation by G. Tintner and M.V. Rama Sastry.

VIII. Application of the H-formula to <u>interregional economics</u> was also made (A. Charnes et al.). It was criticized by S. Hansen and by M. Beckmann.

IX. H measures the expected "<u>optimal incentive to forecaster</u>" under conditions (I.J. Good) which, however, were shown to be very special ones (J. McCarthy; A.D. Hendrickson and R.J. Buehler).

X. Finally, not as a mathematical analogy but as a physico -sociological fact, N. Georgescu-Roegen has applied to the <u>environment of industrial societies</u> (as Schroedinger did to that of an organism) the above physical relation between heat supply and entropy increment, and the implied law of increasing entropy. He insists on its nonstochastic, hence more impatiently pessimistic, version.

ABSTRACT

The entropy formula of information theory measures, in the limit, the minimum expected number of symbols needed to store or transmit a long sequence of decodable messages. It is not related to other quantities, also relevant to the cost, or relevant to the benefit, of information. And if it is at all useful to compare degrees of "uncertainty", any concave symmetric function on probability space has the "intuitively" desirable properties.

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All this must be distinguished from the study of physical entropy in organized human agglomerations.

BIBLIOGRAPHY

Beckmann, M., "Entropy, gravity and utility", Information, Inference and Decision, ed. by G. Menges, pp.155-166. D. Reidel, 1974.

Brumat, C., Uses and Abuses of Entropy in Economics. Ph.D. dissertation in progress. University of California at Los Angeles.

Cantelli, F.P., "Sulla deduzione delle leggi di frequeza da considerazioni di probabilita". Metron, 1 (1921), 83-91. C. Brumat has drawn my attention to Cantelli's more recent and broader treatment of the problem in "Sur les difficultes qui se presentent quand on se fonde sur des considerations de probabilites pour en deduire des lois de frequence".. Economie Appliquee, 10 (1957), 151-60.

Charnes, A., W.M. Raike, and C.O. Battinger, "An extremal and information-theoretic characterization of some interzonal transfer models". Socio-Econ. Plan. Sci., 6 (1972), 531-37.

Cozzolino, J.M. and M.J. Zahner, "The maximum-entropy distribution of the future market price of a stock". Operations Research, 21 (1973), 1200-11.

De Groot, M.H., "Uncertainty, information, and sequential experiments". Annals of Math. Stat., 33 (1962), 404-19.

Georgescu-Roegen, N., The Entropy Law and the Economic Progress, Cambridge, Mass.,(1971).

Good, I.J., "Rational Decisions". J. Royal Statistic Soc., B14 (1952), 357-64.

Griffeath, D. and J.L. Snell, "Optimal stopping in the stock market". Annals of Probability, 2 (1974), 1-13.

Hansen, B., "Utility, accessibility and entropy in spatial modelling", Swedish Journal of Economics, 74 (1972), 35-44.

Hart, P.E., "Entropy and other measures of concentration". J. Royal Stat. Soc., A, 134 (1971), 73-85.

Hendrickson, A.D. and R.J. Buehler, "Proper scores for probability forecasters", Annals of Math. Stat., 42 (1971), 1916-21.

Hildenbrand, W. and H. Paschen, "Ein axiomatisch begrundetes Konzentrationsmass". Statist. Inform. Europ. Wirtschafts., 3 (1964), 53-61.

Jaynes, E.T., "Information theory and statistical mechanics", Phys. Review, 106 (1957), 620-30; 108 (1957), 171-91. -- "Prior probabilities", IEEE Transactions on Systems Science and Cybernetics, SSC-4 (1968), 227-41.

Kullback, S., Information Theory and Statistics. New York 1957. -- "The two concepts of information". J. Amer. Stat. Ass., 62 (1967), 685-6.

Marschak, J., "Economics of information systems". Journ. Amer. Stat. Ass., 66 (1971), 192-219. -- "Limited role of entropy in information economics". 5th Conference on Optimization Techniques, Part II, ed. by R. Conti and A. Ruberti, 264-71. Springer Verlag, 1973. -- "Information, Decision, and the Scientist", esp. pp.173-5, in Pragmatic Aspects of Human Communication, ed. by C. Cherry, D. Reidel, 1974.

McCarthy, J., "Measures of the value of information". Proc. Nat. Acad. Sci., 42 (1956), 654-56.

Reiss, H., "Thermodynamic - like transformations in information theory", J. Stat. Phys., 1 (1969); and a personal communication.

Schroedinger, E., Statistical Thermodynamics. Cambridge 1948 -- What is Life? Cambridge 1944.

Theil, H., Economics and Information Theory. New York 1967.

Tintner, G. and M.V. Rama Sastry, "Information theory and the statistical estimation of econometric relations". Multivariate Analysis, Vol. 2, 687-96, New York 1969.

Tribus, M., Rational Descriptions, Decisions and Designs, New York 1969.